Creativity in Music:
Theories, Strategies, and Composition Software

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Abstract. The functorial approach (functors are structure-preserving maps between mathematical categories) to mathematical music theory and music informatics is presented. It is applied to the theory of musical gestures and its implementation in the Rubato software. We discuss musical analyses based upon gestural theory. Then the mathematical approach to musical creativity is introduced. Using Yoneda's lemma these ideas are applied to understand Beethoven's six variations of the third movement of sonata op. 109. The paper concludes with the semiotic generalization of this mathematical model and its application to a theory of contemporary counterpoint.

In this paper we refer to the functorial approach in mathematical music theories (Mazzola, 2002). We shall explain in details what "functorial" means in the next section "Functors in Music". Suffice it here to say that it is based on category theory, a unified approach to modern geometry, logic, computer science, and theoretical physics. A category is a formal structure that describes a determined type of mathematical objects, such as groups, or vector spaces, together with corresponding functions, which preserve the given object structures, such as group homomorphisms, or linear maps. A characteristic result of this theory is Yoneda's lemma. It states in intuitive terms that you can understand a sculpture if you integrate all its perspectives. In music: you understand a composition if you know all its interpre-tations. We give a detailed description of this lemma in the section "Yoneda's Lemma".

This method has been applied to models of harmony, counterpoint, and melody (Mazzola, 2002). It has been implemented in the Rubato software, developed since 1992 at the Computer Science Institute of the University of Zurich. Rubato is documented in (Milmeister, 2009). Let us shortly describe the design concept of Rubato. The software was developed for musical analysis, performance, and composition. Its components are called *rubettes*. They can perform any specific task. For example, the Metrorubette takes a MIDI file and calculates a metrical and rhythmical analysis associated with the file's onsets. There are rubettes for melodic analysis, harmonic analysis, musical composition, and there is a Performancerubette, which calculates a musical performance (tempo, articulation, dynamics) of a MIDI file, using the analytical rubettes' results for this file. Rubettes can be connected with each other in order to pipe output results to input slots. For example, the output of the Metrorubette's metrical analysis is piped to the input of the Performancerubette to shape performance. Rubettes can be connected to build rubette *networks*. The condition of universal connectability among rubettes is that the output format must be compatible with the input format of rubette data. To meet this requirement, we have developed a universal musical data format, whose instances are called *denotators*. Rubettes understand this format and can therefore be interconnected without any limitation.

The functorial approach has also been applied to model musical gestures, and a Rubato module, the BigBang rubette, for gestural composition, has been implemented by Florian Thalmann (Mazzola and Thalmann, 2011). Gestures are systems of continuous curves in topological categories. This theory has also been applied to understand motivic constructions in Beethoven's sonata op. 106 (see Mazzola 2009).

Functoriality is not only a creative tool, but it allows to model musical creativity in general. We first describe a mathematical model of creativity, based on Yoneda's lemma, and then generalize it to a generic semiotic model of creativity. Its principle is based on the exhibition of a creational *open question*, then the determination of its *semiotic context*, together with the specification of a *critical concept*, then the investigation of the concept's problematic properties (which we call *walls*), then their *opening*, then the *extension* of the concept, and the *application* thereof to the given problem.

Both, the mathematical and the semiotic model, are described in detail in the book (Mazzola, Park, and Thalmann 2011). In this paper we give a summary of these theories and some applications and hope that our contributions might inspire music theorists, musicians, and computer music programmers as well.

Functors for Music

We introduced functors in music in (Mazzola 1985) and then implemented the functorial approach systematically in the music software Rubato since 1992. The original motivation to do so was the music-philosophical insight that understanding a musical composition means to look at it from all possible points of view, in other words: to interpret it in all possible ways and then to integrate these perspectives.

The general framework for such an approach was mathematical category theory, invented by Samuel Eilenberg and Saunders Mac Lane around 1945 to better understand some universal constructions in algebraic topology. Since then, category theory has become a driving force in many advanced mathematical areas, especially in algebraic geometry, where Alexander Grothendieck and his French school triumphed in solving the high ranked conjectures of Weil and Fermat.

Let us shortly explain category theory. The central concept is that of a category $\mathcal{C}$. This is a collection of so-called *objects*, denoted by X, Y, Z,…. Together with a set $Hom_{\mathcal{C}}(X,Y)$ for every pair X,Y of objects, whose elements *f* are called *morphisms* from X to Y and are denoted by arrows *f*: X ⟶Y. It is further possible to compose morphism *f*: X ⟶Y, *g*: Y ⟶Z, yielding the composed morphism *g.f*: X ⟶Z, and this operation is associative. Finally, we have an identity morphism Id_X: X ⟶X for every object X, such that $f.Id_X = f = Id_Y.f$ for every morphism *f*: X ⟶Y.

The standard example of a category is the category $\mathcal{C}$ = $\mathcal{Sets}$ of sets with the sets as objects and the set maps as morphisms. Other well-known categories are $\mathcal{C}$ = $\mathcal{Top}$, the category of topological spaces and continuous functions, or $\mathcal{C}$ = $\mathcal{Dig}$, the category of directed graphs (digraphs) and graph maps, or $\mathcal{C}$ = $\mathcal{Mod}_R$, the category of R-modules and affine maps for a commutative ring R.

Unfortunately, most categories are much less natural. The generality of this theory had to be paid by a dramatic loss of intuitive power. The Japanese computer scientist Nobuo Yoneda found a solution to this problem in 1955, a theorem called the Yoneda lemma. It is easy to prove but has enormous consequences since it allows to reinterpret objects and morphism in terms of classical sets and set maps, even in the most abstract categories.

Yoneda only used one more concept from category theory to present his result, namely the concept of a functor. A functor is a kind of map between two categories. More precisely, if $\mathcal{C}$, $\mathcal{D}$ are two categories, a (contravariant) functor is a map t: $\mathcal{C}$ ⟶ $\mathcal{D}$ that maps every object X of $\mathcal{C}$ to an object t(X) of $\mathcal{D}$, and also maps the sets $Hom_{\mathcal{C}}(X,Y)$ to the sets $Hom_{\mathcal{D}}(t(Y),t(X))$ (attention: reversing arrow directions, this means contravariant) in such a way that the identities go to identities and such that the composition of morphisms goes to the (reversed) composition of their images: t(*g.f*) = t(*f*) . t(*g*). Yoneda looks a very special such functors. He selects an object X of $\mathcal{C}$ and then defines the functor @X: $\mathcal{C}$ ⟶ $\mathcal{Sets}$, mapping an object Y of $\mathcal{C}$ to the set Y@X = $Hom_{\mathcal{C}}(Y,X)$, and mapping a morphism *f*: Y ⟶Z to the set map *f*@X: Z@X ⟶ Y@X with *f*@X(g) = *g.f*. This means that Yoneda replaced every abstract object X of the category $\mathcal{C}$ by a system of sets Y@X, parametrized by the objects Y of $\mathcal{C}$, and connected by the set maps *f*@X. Instead of looking at the abstract object X Yoneda suggested to look at the sets Y@X. Intuitively, this means to look at all the arrows (morphisms) *g*: Y ⟶X that start at object Y and target at object X. You imagine sitting on Y and looking at X. This is why Y is also called the *address* of *g*. Instead of sets, general category theory envisages systems of sets Y@X that are parametrized by the addresses Y, objects of $\mathcal{C}$. Elements of Y@X are called Y-*addressed points of* X.

Yoneda's Lemma

What is the big advantage of Yoneda's approach? He could prove that if you know the functor @X, you know all of X! Without delving into technical details we may just acknowledge that Yoneda's lemma proves that two objects X_1 and X_2 are isomorphic if and only if their functors $@X_1$ and $@X_2$ are so. Intuitively this means that *understanding* X *is equivalent to understanding its functor* @X, and this in turn means to understand all the sets Y@X with variable addresses Y.

This result was a dramatic step towards a better control of abstract categories by a reconstruction of classical sets and set maps!

Although it seems that the functor @X is much richer than X, it is essentially the same, this is another statement of Yoneda's lemma. We may now start working in @X instead of X and do many useful and essential operations which in fact are all restatements of facts that regard X. In mathematical music theory, we systematically used this functorial point of view to define and analyze musical structures and relations. Let us look at some representative examples.

Lewin's Time Spaces. David Lewin uses time-related structures, time spans. A time span is a pair (b,x) of an onset time b and a (non-zero) duration x, both are real numbers. The analysis of Lewin's transformational laws between such time spans (Mazzola, 2002, p. 83) shows that these objects are precisely the $\mathbb{R}$-addressed points (b,x): $\mathbb{R}$ ⟶ **Onset**($\mathbb{R}$), with (b,x)(t) = b + xt, of the one-dimensional real time space **Onset**($\mathbb{R}$).

Dodecaphonic Rows. A dodecaphonic row r in the pitch class space **PitchClass**($\mathbb{Z}_{12}$) is a sequence $r = (r_0, r_1, \ldots r_{11})$ of pairwise different pitch classes r_i in **PitchClass**($\mathbb{Z}_{12}$). This is equivalent to giving an $\mathbb{Z}^{11}$-addressed point r: $\mathbb{Z}^{11} \longrightarrow$ **PitchClass**($\mathbb{Z}_{12}$). More generally, a row in *serial theory* is an $\mathbb{Z}^{11}$-addressed point r: $\mathbb{Z}^{11} \longrightarrow$ **P** of a parameter space **P** (with twelve pairwise different values) that could represent duration, loudness, attack type, etc. For details, see our analysis of Boulez's serial composition *Structures pour deux pianos* (Mazzola, Losada, Thalmann, and Tsuda, 2009). It is remarkable that the entire compositional approach by Boulez resides on operations on address $\mathbb{Z}^{11}$ as well as on its derived address, the affine tensor product $\mathbb{Z}^{11} \boxplus \mathbb{Z}^{11}$.

Harmony. In his reconstruction of Riemannian harmony, Thomas Noll (Noll, 1995) uses self-addressed points p of pitch classes, i.e. points p: $\mathbb{Z}_{12} \longrightarrow$ **PitchClass**($\mathbb{Z}_{12}$) instead of pitch classes. Pitch classes then are interpreted as constant self-addressed points.

Counterpoint. The mathematical theory of counterpoint (Mazzola, 2002, Part VII) views intervals of pitch classes as points i: $\mathbb{Z} \longrightarrow$ **PitchClass**($\mathbb{Z}_{12}$), which is equivalent to give two pitch classes: one, namely cf(i) = i(0), for the cantus firmus value, and the other, namely d(i) = i(1)-i(0), for the interval value from cantus firmus to discantus. The theory then uses this identification of an interval i with a pair (cf(i), d(i)) to calculate contrapuntal structures and laws. The algebra for this calculus is provided by the representation of the pair (cf(i), d(i)) as a dual number cf(i)+ ε.d(i) in $\mathbb{Z}_{12}[\varepsilon]$ ($\varepsilon^2 = 0$).

Gestures: A Musical String Theory

Despite the power of functors, mathematical music theory still lacked the description and analysis of an essential aspect of musical realm: gestures. Already David Lewin in his celebrated book about musical transformations (Lewin 1987) speaks about gestures. However, his theory is still processual, half way between facts and gestures in terms of the dimension of embodiment. But it is remarkable that, relating to his crucial question about the transformational movement from a musical point s to a point t, he adds: "This attitude is by and large the attitude of someone inside the music, as idealized dancer and/or singer. No external observer (analyst, listener) is needed." Lewin felt that gestures are the substance underlying processual transformations. But he was not well grounded in the mathematical theories for gestural approaches.

In (Mazzola and Andreatta 2007) and (Mazzola 2009) we have exposed a mathemaical theory of musical gestures, also called a musical string theory because it essentially relies on continuous curves in topological spaces, similar to physical string theory. A gesture in a topological space X is defined as follows. We consider the digraph Curves(X) whose arrows are the continuous curves c: I $\longrightarrow$ X, defined on the real unit interval I = [0,1], and whose vertices are the points of X, while the head of an arrow c is h(c) = c(1), and its tail is t(c) = c(0). A gesture in X is a digraph morphism g: Γ $\longrightarrow$ Curves(X), where the digraph Γ is called the curve's *skeleton*, and X is called its *body*. This means that every arrow in the skeleton is associated with a continuous curve in X, and the coincidence of heads and tails of arrows in the skeleton is also met in the body, see Figure 1.

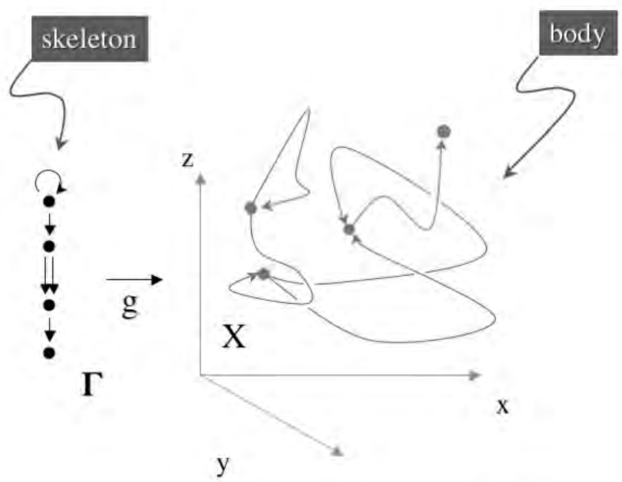

Figure 1. A gesture g with skeleton Γ and body X.

Although gestures look like curve structures, they actually enable very complex shapes by the following fact: The set Γ@Curves(X) of all gestures with skeleton Γ and body X is canonically provided with a topology, i.e. it is itself a topological space. It therefore makes sense to consider gestures h: Δ $\longrightarrow$ Curves(Γ@Curves(X)) of gestures! We call such gestures *hypergestures*. And this process can be iterated *ad libitum*. If we have a sequence $\Gamma_n, \Gamma_{n-1}, \Gamma_{n-2}, \ldots \Gamma_1$ of digraphs, we denote by $\Gamma_n@\Gamma_{n-1}@\Gamma_{n-2}@\ldots \Gamma_1@$ Curves(X) the space of n-fold hypergestures. It can be shown that this space is topologically isomorphic (homeomorphic) to any space $\Gamma_{p(n)}@\Gamma_{p(n-1)}@\Gamma_{p(n-2)}@\ldots \Gamma_{p(1)}@$ Curves(X) constructed by a permutation p of the n digraphs. This theorem is called the *Escher Theorem* (Mazzola and Andreatta 2007) since it allows to view internal digraphs in hypergestures as external ones and vice versa, similar to Escher's famous flip-flop graphics.

This enables the construction of hypergestures that are based upon curves of curves of curves, etc. And this means: Gestures that describe surfaces, cubes, and higher voluminous shapes. Moreover, since the body of a gesture can also have time in its coordinates, it is automatically possible to consider gestures that in fact describe *movements of complex bodies in time*. And this is what one expects from a valid theory of gestures: the conceptualization of complex shape movements. We have applied gesture theory for both, musical analysis and composition.

Remark for mathematicians. We should recall the natural generalization which gestures provide for classical algebraic methods. In abstract algebra, diagrams are most important formal setups for complex processes. A diagram is a digraph morphism $g: \Gamma \longrightarrow |\mathcal{C}|$ with values in the digraph $|\mathcal{C}|$ defined by a category $\mathcal{C}$. This can also be restated as a (covariant) functor $g: Path(\Gamma) \longrightarrow \mathcal{C}$ on the path category of digraph Γ. One might then consider the category CPath(Γ) of continuous paths over Γ. It not only looks at paths starting and ending of vertexes of Γ, but takes paths startig and ending anywhere on the arrow shapfts of Γ. This category is called *category of continuous paths* over Γ. One might then look at continuous functors $g: CPath(\Gamma) \longrightarrow \mathcal{C}$ with values in a topological category $\mathcal{C}$. There is an adjointness isomorphism

$$Hom(CPath(\Gamma), \mathcal{C}) \xrightarrow{\sim} Hom(\Gamma, Curves(\mathcal{C}))$$

Here the curves are the continuous covariant functors $I \longrightarrow \mathcal{C}$ from the category I whose obects are the points of I, and whose morphisms are the pairs (x,y), $x \leqq y$. It means that gestures in a topological category $\mathcal{C}$ (in particular in a topological space) are the same as "continuous" diagrams, i.e. diagrams on continuous paths over Γ. Because of the above adjunction, we may call them gesture diagrams, they correspond one-to-one to gestures. Traditional diagrams are obtaind from gesture diagrams by their restritcion to the sub-category $Path(\Gamma) \subset CPath(\Gamma)$.

Gestures in Music Analysis

Let us first discuss a gestural analysis of the beautiful fanfare at the beginning of the Allegro movement in Beethoven's sonata op. 106 (Hammerklavier), see Figure 2. If we represent the time coordinates, onset and duration of these notes, we obtain a set of eight points in the plane spanned by onset as horizontal axis and duration as vertical axis. See Figure 3. This image is a repetition of a slow down movement for the left half. We have drawn it as two gestures from the short to the longer point. The right four point set shows a different construction: Here we have a repetition of eight notes, and then a slowed down version of this repetition. We also have connected the repeted notes as a line gesture. So the first four and the last four points are reversing the construction method: first we have the repetition of a slowing down gesture, second we have a slowing down of a repetition gesture.

Figure 2. The fanfare heading the Allegro movement of Beethoven's sonata op. 106.

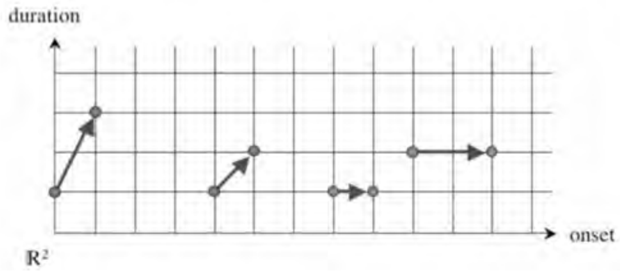

Figure 3. The representation of the fanfare in the onset-duration plane shows two groups of four points each. To the left we have the repetition of a slowing down gesture, to the right we have a slowing down of a repetition gesture.

The question is how these two groups are related as gestures. To make this question precise, we consider the hypergesture ϱ that connects the first (left) slow down gesture to the second slow down gesture. Then we consider the analogue hypergesture σ connecting the first (right) repetition gesture with the second repetition gesture, see Figure 4.

We can connect ϱ to σ by two hyper-hypergestures. The first one is shown with hypergesture ϱ'. We first rotate ϱ and then reverse the direction of the hypergestural parameter, thereby moving from the lower to the higher arrow. Then we deform this hypergesture to σ. This is a problematic action since we have to change the hypergestural parametrization, which is an unnatural mathematical action. The second variant is more natural: We first make a rotation of ϱ around an axis in the onset-duration plane that has direction half angle between horizontal and angle of the first slow down arrow. The resulting hypergesture ϱ'' can be deformed into σ like in the first variant.

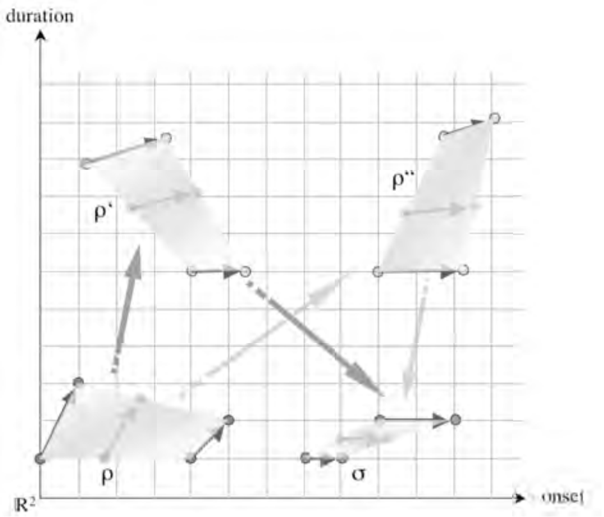

Figure 4. Two hyperhypergestures connecting ϱ with σ.

This second variant also exchanges duraton and onset, and this is what we expect from the former insight that first we have the repetition of a slowing down gesture, second we have a slowing down of a repetition gesture. This reversion is geometrically realized by the rotation around that skew axis. We however observe that this rotation introduces a third dimension. It is the well-known trick that avoids travelling through a mirror: just leaf the figure through the third dimension.

A Composition Software with Finger Gestures

If analysis of compositions can help understand them so intuitively, then one expects that making compositions should also be supported by gestural devices. This in fact an idea that (among other teams) our research team at the University of Minnesota has investigated since 2007 and which has been realized by Florian Thalmann's BigBang rubette (Mazzola and Thalmann, 2011), a module of the universal music software Rubato Composer (Milmeister 2009). The BigBang rubette has been used to compose an orchestral variant of Boulez's famous serial composition *Structures pour deux pianos* (Mazzola, Losada, Thalmann, and Tsuda 2009).

The BigBang rubette uses the following principles to shape a musical composition:

1. It displays a two-dimensional plane (in fact the computer screen, see Figure 5) whose points have two out of five musical coordinates: onset, pitch, loudness, duration, and voice. The user can choose which two coordinates should be shown.
2. The user can load a musical compostion from a MIDI file or draw a set of points on the plane.
3. Apart from Boolean operations on such sets on note-points, the user can operate affine transformations by three-finger gestures on the trackpad. We shall come back to this point soon.
4. The user may change the plane at any time and show the composition from another parametric perspective. The affine transformations of Boolean operations can be continued on this new plane.
5. The user can also create a grid of affine transformations, like a translation grid for ornaments, but with any affine transformation in the grid arrows!
6. The user can also make an alteration, that is a deformation of the composition according to certain points of attraction which act as gravitational forces. This operation is a vast generalization of quantization or pitch alteration known from classical music theory.

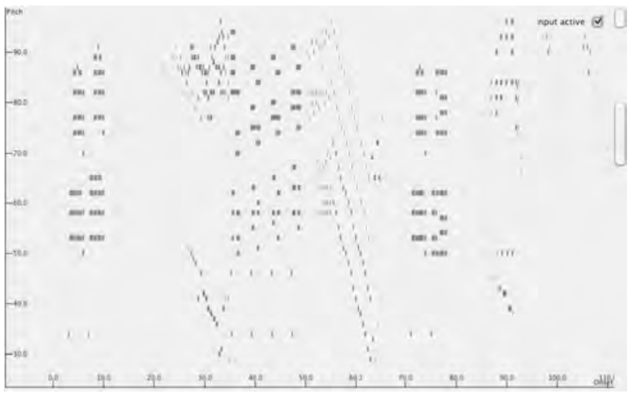

Figure 5. The BigBang Interface showing the first measures of Beethoven's op. 106 in the onset-pitch plane. The rectangles correspond to notes, their position, geometry and color correspond to their musical parameters of onset, pitch, loudness, duration, and voice.

We should comment on the question of the two-dimensional space where all these operations take place. This apparent restriction is much less serious than one would argue since it is a classical fact of affine algebra that any affine transformation in the n-dimensional space can be written as composition of two-dimensional transformations, i.e. transformations that affect only two of the n coordinates. So our user interface allows for n–dimensional transformations.

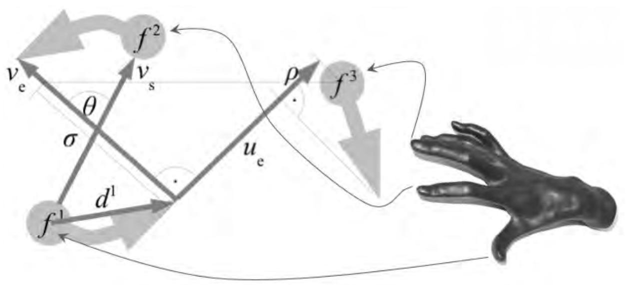

Figure 6. The three-finger gestures represent all possible two-dimensional affine transformations.

Let us show how the three-finger gestures can represent two-dimensional affine transformations, see Figure 6. The movements of the three fingers, starting at positions f^1, f^2, f^3, defines three vectors that give rise to a number of geometric variables, and these have a meaning in terms of classical generators of affine two-dimensional transformations as follows:

1. d_1 ~ translation
2. $|v_e|/|v_s|$ ~ dilation
3. θ ~ rotation
4. σ ~ shearing
5. ϱ ~ reflection

Yoneda's Lemma for Creativity

As we have shown above, the functorial approach with Yoneda's lemma as an outstanding theorem provides excellent technical tools for music theory, musical gestures, and composition. It is natural to ask for deeper reasons for the creative power of Yoneda's lemma.

In (Mazzola, Park, and Thalmann 2011), we have given a theoretical basis to creativity as it may emerge from the thought process induced by Yoneda's lemma. In this paper we want to give a summary of that approach and illustrate it with a discussion of the variational strategies of Beethoven in the third movement of his piano sonata op 109. This subject is presented in detail in (Mazzola, Park, and Thalmann 2011, chapters 19.2 and 26).

Let us first interpret Yoneda's lemma as a mathematical model for creativity. Whenever one is dealing with creativity, one has an open question that is set up in a specific context. The question then focuses on a critical concept in that context. The concept must be inspected for its sensitive properties, which we call "walls". The proper creative action consists in an opening of these walls and then in their extensions to new perspectives, which hopefully yields an answer to the initial open question. Yoneda's lemma deals exactly with such a process. One starts with an open mathematical problem in the context of a specific category $\mathcal{C}$. The problem can usually be related to a critical object X, which one would like to understand. Yoneda's lemma tells us that in order to do so, one has to look the system $\mathcal{D}$ of all perspectives f: V ⟶ X from addresses V in $\mathcal{C}$.

In theory one then integrates these perspectives (together with the address changes over X, i.e. the morphisms between the addresses that commute with the perspectives) and creates the colimit colim($\mathcal{D}$)⟶ X. This is a kind of union of all addresses with their perspectives to X, patched together by the relations given by address changes, see (Mac Lane 1971) for technical details. This huge space is of no usage in practice. One rather looks for a minimal subset of addresses and perspectives which characterizes X. For example, if we work in the category Sets of sets, it is sufficient to look at all *element* perspectives e: {1} ⟶ X from the singelton set {1} since their number is exactly the cardinality of X, and this is enough to know all about X in set theory!

So let us suppose we have found a characteristic system $\mathcal{D}_{char}$ of perspectives f_i: V_i ⟶ X. This is what we could interpret as being the relevant walls of X. They must be understood in the view they generate upon X. We then step over to their colimit and its integrated perspective f: colim($\mathcal{D}_{char}$)⟶ X upon X. This colimit is a kind of extension of the wall system of X. In order to understand X one is asked to investigate colim($\mathcal{D}_{char}$) and to extend one's understanding of X to that of this colimit. This enriched view upon X is the core of creativity in this mathematical environment[1].

Beethoven's Six Variations in Sonata op. 109

Let us give a musical illustration of this creative process with Beethoven's six variations of the third movement of piano sonata op 109. In this discussion, we follow the brilliant analysis given by Jürgen Uhde (Uhde 1974). The third movement is a sequence of six variations of the main theme X, which is entitled "Lyrical, with deepest sentiment", see Figure 7.

Figure 7. The main theme X of Beethoven's sonata op. 109.

Uhde describes the six variations in a geometric way as perspectives onto the theme. So let us think of them as being morphisms f_i: V_i ⟶ X, i = 1,2,3,4,5,6, although we do not specify a precise mathematical category here. Each of these perspectives stresses a particular aspect of X. When the first five perspectives are over Uhde asks whether there is still a reasonable positon for a sixth variation and adds: “Wasn't the theme illuminated form all sides from near and from far, and following sound and structure? The preceding variations `danced' around the theme, and each was devoted to another thematic property. “ Each of these perspectives focuses on a specific aspect, on (1) melody, (2) rhythm, (3) counterpoint, (4) permutation, and (5) third intervals. These are the characteristic perspectives. And we would expect from our above Yoneda-driven model of creativity that their colimit should be built. This colimit would then constitute a concise perspective onto the main theme X, containing all the previous insights in a single item. It is natural to interpret the sixth variation as being this colimit. And Uhde gives a fascinating interpretation of the sixth variation. He views it as if it were itself a body of six micro-variations, and he describes this body as a “streamland with bridges”, the bridges connecting the six micro-variations. This is very similar to the construction of a colimit, which is also essentially a landscape connecting its components by bridge functions.

Looking at the sixth variation it in fact contains a number of restatements of the theme, but then dramatically converges to a finale that is a kind of synthesis of all these aspects, and Uhde describes this final explosion of energy that is incited by the dominant's vibrational axis with that long lasting trill b - c♯ It is however critical that the dominant lasts so long (measure 165-187, almost half of the sixth variation!) without being resolved to the tonic. Expecting the cadential function of the dominant, the audience would be annoyed when hearing such a never ending announcement of the tonic. The function of this trill must be different. We can make this explosion more precise.

The dominant's vibration is more concretely shown as a alternative rendition of the dominant tone b and the neighboring c♯, see Figure 8. This rotation around these two tones is first set as a set of explicit notes, but then with increasing energy converges to a fulminant final explosion of a trill. William Kinderman, in his description of the finale of op. 109 (Kinderman 2003) writes: “Through a kind of radioactive breakup, the theme virtually explodes from within, yielding an array of shimmering, vibrating sounds. One might regard the sustained trill on B with its upper neighbor C♯, which migrates into the treble in the passage before the thematic reprise, as the utmost elaboration of the melodic peak of the sarabande on these two notes.”

Figure 8. The beginning of the trill explosion in measure 164 of the third movement of op. 109.

This trill pairing is not only a nice emotional effect of rotational energy, but it has a very substantial interpretation in terms of a cadence of the underlying tonality! In fact, the inversional movement $I_{b/c\sharp}$ around b, c♯ is identical to the unique inversion of the E-major scale of this piece, in other words, the trill characterizes or cadences the basic tonality via its unique inner symmetry inversion. This final trill explosion is a big cadence of the piece’s tonality E-major. The presence of this inverson is also strongly evident from inversion-symmetric intervallic movements in the sixth variation.

Summarizing, the sixth variation is very similar to a colimit of the first five variations and as such gives a concise and characteristic synopsis of these varied perspectives, culminating in a cadential trill explosion for the piece’s tonality E-major.

A Semiotic Model of Creativity

In view of the mathematically motivated model of creativity, we have developed a more generic model in order to obtain a scheme of creative processes that can be applied to musical situations that are not related to mathematical paradigms. And we also studied relevant literature about creativity and learned that many historical and present models of creativity are strongly related to psychological or cognitive aspects. These studies are reported and commented in (Mazzola, Park, and Thalmann 2011). In view of our Yoneda-oriented model we had in mind some approach to creativity that was more explicit as a process scheme, operational, and not based upon psychological aspects.

We further wanted to demystify the concept of creativity in the sense of a belief that creativity is not in whatever sense an inspirational, divine or otherwise otherworldly grounded miraculous phenomenon. Of course there is no guarantee that one becomes creative in every case, but we are persuaded that there is a strong rational component in creativity, and this is what we have attempted to model.

Following the above mathematical approach, we propose to view creativity as being initiated by an *open question*. Ideally, the creative process will provide us with an answer to this question, and this will be the final step in our model: to test our efforts in view of an answer to that initial question. The open question must be asked within a determined context. Creativity need a system of reference since some achievement might be creative in one context, but not in another. For example, a monkey might be creative in solving a problem of how to grasp a banana in a difficult position, while the same achievement is not creative at all for a human. We have chosen such a reference system to be a semiotic, a sign system. This is important since we argue that creativity must always be an extension of a system by added contents. Purely formal extensions cannot be qualified as being creative. For example, the set of all sonatas is, in the formal system of musical data formats, given as a point in a big powerset. But Beethoven's composition sonata op. 57 (Appassionata) is a creative achievement not because of its membership in that big powerset, but because Beethoven added significant contents to what is a sonata in the semiotics of musical compositions.

Next, relating to the initial open question, one is asked to identify a *critical concept* (sign) in the semiotic context, a concept that might be crucial to answer the initial question.

Looking at this critical concept, we propose to look for its properties or attributes which are responsible for the critical status of the concept. We call these properties the concept's *walls*. In Einstein's solution of the problem of time in special relativity, the wall was the singular of time: There was essentially only the divine Newtonian time. Such a wall can be very difficult to be identified. Perhaps this is the hard point in a creative process. Some walls are so near that you might not become aware of there mere exisence. That there might be more than one time is such a wall.

The next step is that one would try to soften the walls, to find out where they could be eliminated or at least opened. In Einstein's situation the question of why time must be a singular was such an opening action.

Next, one is asked to extend the space beyond the opened wall in a constructive way. In Einstein's case, this was the question of how to deal with a multiplicity of times. And his solution was the Lorentz transformation that connected space and time for inertial systems in relative motion. Recall that such extesions will always essentially be enrichments of the semiotic system, an added value in the signs' contents, not just forms.

The process terminates with a test phase to find out whether and how far the open question has been solved by the proposed extensions of the critical concept's walls.

Creating Contemporary Counterpoint

Let us conclude this paper with an example of a creative process in music that is related to the theory of counterpoint, see also (Mazzola, Park, and Thalmann 2011, chapter 11).

Counterpoint is perhaps the most critical topic when it comes to search for creativity. In fact, counterpoint is strictly codified and taught as a sort of catechism of classical compositional discipline in polyphony, the art of combining several voices to a balance interplay. In its most formalized shape, counterpoint has been described in Johann Joseph Fux's famous treatise *Gradus ad Parnassum* (Fux 1725). It was written in Bach's time but consciously refers to Fux's idol Giovanni Pierluigi da Palestrina. In the foreword, Fux even stresses his conservative position: "Why should I be doing so (write about music) just at this time when music has become almost arbitrary and composers refuse to be bound by any rules and principle, detesting the very name of school and law like death itself." This last sentence could have been written in our time where everything seems to be possible. Our challenge will be to inspect the deep structure of this theory in order to discover its inherent potential for a creative extension.

In view of the role of counterpoint pedagogy in music schools, this seems an important enterprise. Counterpoint is commonly understood as a frozen knowledge that can at best serve as a gymnastic exercise in compositional rigor, but never as a tool that one would actually use for contemporary composition.

The open question. Let us get off ground with the simplest situation in Fuxian counterpoint, see Figure 9. We are given a melody line, called cantus firmus, c.f. Classically, it is composed according to the style of Gregorian chant, but this is not important for our concerns. The contrapuntal assignment is to invent a second melody, called discantus to be added to c.f. and fulfilling certain rules. In our example, we have the simplest case, called first species, where over each c.f. note we have to position a discantus note of same duration. The contrapuntal rules give us constraints of how to position these notes.

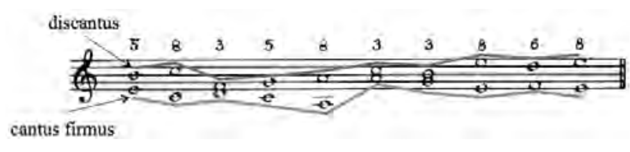

Figure 9. The contrapuntal assignment is to invent a second melody, called discantus to be added to cantus firmus (c.f.) and fulfilling certain rules.

There are two parts of this rule system. The first part is completely rigid: It allows us to set a discantus note over the c.f. note only if the interval (number of semitone steps) from the c.f. to the discantus note is a consonant interval. This means that this consonance can be either a prime (or unision) (0 steps, same pitch), a minor third (3 semitones), a major third (4 semitones), a perfect fifth (7 semitones), a minor sixth (8 semitones), or a major sixth (9 semitones). We may also use intervals derived from these when adding an octave (12 semitones), for example the octave (0+12 = 12 semitones), the minor tenth (3+12 = 15 semitones), etc. All other intervals are called dissonances, and they are strictly forbidden. In our example you see numbers above each interval, they are just the steps in the C-major scale, i.e. 5 stands for the fifth (the fifth note above e), 8 for the octave (the eight note above c), etc.

The second part of contrapuntal rules prescribes which intervals may succeed which intervals. In this rule system, one distinguishes between perfect consonances (prime, octave, fifth) and imperfect consonances (thirds, sixths). Further, different movements form one interval to the next are conceived: Direct motion, i.e. both voices ascend or descend. Contrary motion: one voice ascends while the other descends, or vice versa. Oblique motion: One voice moves while the other remains on the same pitch. The fundamental rules as stated by Fux can be summarized in one single rule: Any motion to a perfect consonance must be contrary or oblique. In particular the famous parallels of fifths are forbidden, since they are direct motion to the perfect fifth consonance.

These rules are very formal, but they intend to produce a double gesture of cantus firmus with discantus that alternates between perfect and imperfect intervals. In fact, once you are at a perfect interval (prime, octave, perfect fifth), there is not much of a choice to move to another perfect interval, therefore you want to go to an imperfect interval, where the choice is larger. But again, these formal properties are not what we experience in music, it is rather a well-shaped pairing of voices that we hear and one that has a dynamical power. The rules seem to be the formal result of a principle of interacting forces. But what is the force guiding this interaction? It is somehow related to the inner life of consonances, but it is mysterious how this happens.

Our open question therefore might be this: What are the forces that drive the contrapuntal movement of voices?

The context. The context of our question is multifaceted. On the one hand we recognize the melodic context: Counterpoint deals with a multiplicity of melodic voices. We have see the cantus firmus and discantus voices here, but the general setup deals with three four or more voices. Second, the intervallic aspect refers to harmonic considerations. Consonances and dissonances are harmonic functions on intervals. They are not Riemannian (no T, D, S values here), but have two values κ (for consonant) and δ (for dissonant). We do not consider more complex chords, but only intervals. And it is a strict theory that attributes to six intervals exactly one value.

And that value is also invariant under octave extension, i.e. consonances are defined on pitch classes modulo octave, this is the so-called octave equivalence. We might therefore define the set of K = {0, 3, 4, 7, 8, 9} of consonant intervals (of pitch classes), whereas the complement D = {1, 2, 5, 6, 10, 11} contains the dissonances. We use here the domain $\mathbb{Z}_{12}$ of the twelve chromatic pitch classes with semitone steps, which might be best represented by the common clock circle that has twelve hours (0, 1, 2, 3, ... 11, 12=0), and where every hour represents a key on the piano, up to octave. Then K ∪ D = $\mathbb{Z}_{12}$ and have no interval in common: K ∩ D = ∅, see Figure 10.

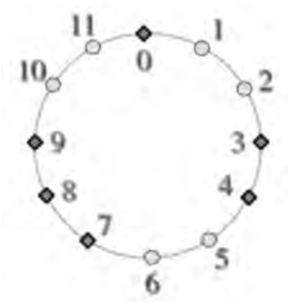

Figure 10. The sets K (squares) and D (circles) of consonances and dissonances. They cover the set of all twelve intervals of pitch classes, contain each six intervals and have no common elements.

This setup in counterpoint is quite exceptional since the perfect fifth (7 semitones) is consonant as we expect it from Pythagorean tradition, but the perfect fourth (5 semitones) is dissonant, which opposes to the Pythagorean tradition and to the acoustical theories about consonance altogether. Counterpoint is not an acoustical theory, the simple frequency ratio 4:3 for perfect fourth is not an argument for consonance. We learn that counterpoint is a highly symbolic theory that is independent of acoustical reality. It is a construction for the sake of composition of melodies, not for frequency ratios.

The admitted consonant intervals are a kind of gestural movements from c.f. to discantus, and the entire discantus melody could be understood as the final position of a big sweeping gesture starting at the c.f. melody. The common understanding of counterpoint is that its Latin etymology *punctus contra punctum*, point against point, means the interval's note 'points' as opposed to each other. The discantus is then viewed as a 'vibrational' deformation of the c.f. It is straightforward that such a vibration requires the consideration of the forces of 'elasticity' which relate c.f. to discantus.

The critical concept. In music, the movement of voices and their interplay expresses a dynamical exchange of forces that create the interesting connections of parts not as static entities, but as poles spanning and being spanned by a force field. This is a remarkable phenomenon in musical syntax: Although its components look like static structures, they are in fact the visible part of an entire system of tension and relaxation.

We therefore suggest to focus on the critical concept of a contrapuntal tension and relaxation[2]*.*

Inspect the concept's walls! When we talk about tension, the first question that arises is about its direction: between what and what is the tension spanned? The first answer might seem to be obvious: It is spanned between the two voice gestures, and this is what common understanding of counterpoint suggests. But then, here is a first wall: why should this be a tension? We have only consonances that move the c.f. to its discantus. What kind of tension can this be?

The second wall is about the nature of such a tension. What is creating the force between the tensed parts? It is also a wall since in view of the first wall, we do not see any kind of tension within the consonances.

Soften and open the walls!

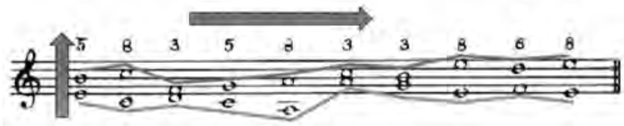

Figure 11. Two interpretations of counterpoint as a hypergesture. The first is a hypergesture of melodic lines from c.f. to discantus. The second, a hypergesture of intervals, is more adequate to the original contrapuntal thinking.

The first wall looks quite hard. The only direction that is visible is the gestural movement from c.f. to discantus. This is due to the fact that the classical construction of counterpoint gets off ground after the c.f. melody has been built. But let us look more carefully at a composed counterpoint in the sense of a gestural movement. We have the c.f. gesture and this one is taken as a starting point of a gestural movement towards the discantus gesture. But if we look more precisely, this "hypergestural" construction (gesture of gestures) consists in a sequence of tiny intervallic gestures. The melodic gesture is composed of many intervallic gestures. We first build the c.f. gesture and then successively move in each interval to the cantus firmus gesture, see Figure 11, where this movement is indicated by a vertical arrow.

But this is not mandatory. One may also view the resulting counterpoint as being generated by a hypergesture that moves horizontally from left to right. And this one would then take each interval gesture and move it to the next to the right. This alternative is a very simple special case of what in gesture theory is known as the Escher theorem: One may always exchange the order of generating gestural movements, see our remark for a detailed account on this far-reaching principle.

Is this reinterpretation musically reasonable? Historically speaking it is since it has been shown that the concept of punctus contra punctum was not the vertical opposition of voices, but the opposition of successive intervals. In other words, the concept of a point (punctus) in this contrapuntal approach was the entire intervals! Each interval is thought of being a kind of 'thick' point, a point in the contrapuntal space. This is completely natural. The contrapuntal idea takes intervals and moves them around. This is not in contradiction to the Gregorian construction of the c.f line, but it tells us that the tension has another direction: The movement proceeds from interval to interval, although the intervals are all germinating from the c.f. notes.

Does this opening help approach the second wall: the nature of the contrapuntal tension and relaxation? The question now is no longer about a tension between c.f. and discantus notes, but between successive consonant intervals. We are looking at the set of consonant intervals and move around in this set. But can there be an kind of tension on this seemingly compact set of consonant intervals. After all they are all consonant. This is true, but we know that there are intervals which are more consonant than others: Some are perfect, the others are imperfect. It is interesting that in the historical development of contrapuntal theories, imperfect intervals have also been called dissonant, see [102] for a detailed account on this terminological finesse in the history of counterpoint! The question about this second wall would then be whether the set of consonant interval bears something dissonant, whether some intervals might be infected by a dissonant character, and then moving from perfect to imperfect intervals would be a hidden change from consonances to dissonances.

Extending open walls. The opening of the first wall refers to what can be done for the second wall. We are confronted with the set K of consonant intervals. We are searching for forces that might explain the tension between successive consonant intervals. The next question is therefore about the nature of such a tension.

Evidently this is a deep problem since it regards the very definition of consonances and dissonances. How are they characterized? Is there any chance to conceive of such a tension from a deeper understanding of the dichotomy of intervals defined by splitting them into the complementary sets K, D?

We have already seen that there is no acoustically valid definition of consonances in counterpoint since the perfect fourth is dissonant, whereas the perfect fifth is consonant. Also observe that no acoustical theory of consonances would allow for a strict separation of consonances from dissonances, there are only degrees of consonance in acoustical theories. Let us look how these theories would define consonances. They would define a function (like Euler's gradus suavitatis function) that would be evaluated for each interval and yield more or less high 'sonance' values. But this is not the only way to define K! One may also define the whole set instead of first collecting its members. There is in fact an interesting way to do so and to arrive at the set K.

The solution looks as follows: We do look at all splittings of the ensemble $\mathbb{Z}_{12}$ of all intervals (of pitch classes) into two six-element complementary sets (X,Y), i.e. $X \cup Y = \mathbb{Z}_{12}$ and $X \cap Y = \emptyset$. But how to find exactly our candidate (K,D)? We have a first very important property of this splitting, namely that there is exactly one symmetry that maps each element of K to an element of D, and vice versa, and this symmetry is $A(x) = 5x + 2$, it is called *autocomplementarity* symmetry.

In Mathematical Music Theory (Mazzola 2002) it has been shown that there are essentially only six such interval dichotomies, they are called strong dichotomies. Here they are, together with their unique autocomplementarity symmetry, their number relates to the classification theory in (Mazzola 2002, chapter 30):

1. Nr. 64, $\Delta_{64} = (I,J) = (\{2,4,5,7,9,11\},\{0,1,3,6,8,10\})$,
   $A_{64}(x) = 11x + 5$
2. Nr. 68, $\Delta_{68} = (\{0,1,2,3,5,8\},\{4,6,7,9,10,11\})$,
   $A_{68}(x) = 5x + 6$
3. Nr. 71, $\Delta_{71} = (\{0,1,2,3,6,7\},\{4,5,8,9,10,11\})$,
   $A_{71}(x) = 11x + 11$
4. Nr. 75, $\Delta_{75} = (\{0,1,2,4,5,8\},\{3,6,7,9,10,11\})$,
   $A_{75}(x) = 11x + 11$
5. Nr. 78, $\Delta_{78} = (\{0,1,2,4,6,10\},\{3,5,7,8,9,11\})$,
   $A_{78}(x) = 11x + 9$
6. Nr. 82, $\Delta_{82} = (K,D) = (\{0,3,4,7,8,9\},\{1,2,5,6,10,11\})$,
   $A_{82}(x) = 5x + 2$

Why do we choose the last one for the classical consonance and dissonance concept of counterpoint? There is a geometric reason: It is the one such that the images A(x) of intervals x have the largest num-ber of third intervals to be connected to each other, this means that A_{82} throws x farther away than every other autocomplementarity function. For example, the fifth $x = 7$ is mapped to $A(x) = 1$, the minor second, which is two minor thirds away. Also the distances among all six consonances in K in terms of number of thirds between any two of them is minimal, they are grouped as tight as possible, but see (Mazzola 2002, chapter 30) for details. Therefore the classical consonance-dissonance dichotomy can be chosen (among the six strong dichotomies) for its geometrically extremal properties.

So we are viewing five other dichotomies on which one could potentially focus. But what is this game helping us understand the nature of contrapuntal tension? We still have not understood how we could import the dichotomy of consonances vs. dissonance into the part K of consonances as suggested by the idea of perfect and imperfect consonances. The situation is really dramatic: On the one hand we have to move from consonance to consonance, on the other we would like to move from consonance to dissonance. This is a plain logical contradiction.

It is as long as we view the entire setup as a configuration of sets of intervals. But we should learn from the very definition of these strong dichotomies that they are effectively defined by their autocomplementarity functions. This is, what we should really take care of. The sets as such are of secondary relevance. Can we try to move from a given consonance ξ to another consonance η as if the latter were a dissonance? This would mean that we have to construct a limiting line between ξ and η that turns the latter into a kind of dissonance.

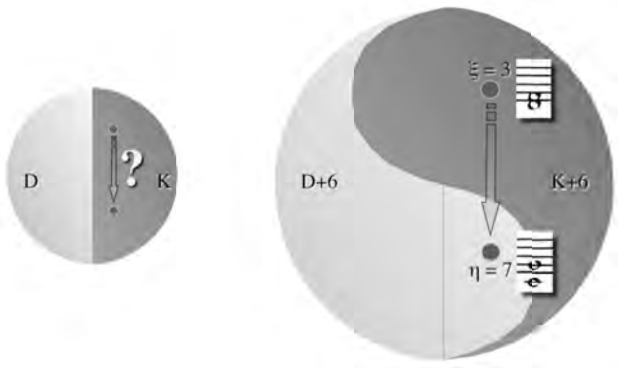

Figure 12. We may move from one consonance to another consonance but move from the 'imaginary' dichotomy's consonant part K + 6 to its dissonant part D + 6.

So why not just transform the consonance-dissonance dichotomy (K,D) such that parts of the D overlap with K? And such that η, but not ξ falls into the transformed part of D? Let us make a picture of such an idea, see Figure 12. To the left you see the two halves K, D covering the interval set. The transition from one consonance to another would be the problem when living inside K. But we could, for example, add a tritone (6 semitones) to K and to D. That would yield $\Delta_{82}+6$ = (K+6,D+6) = ({0+6,3+6,4+6,7+6,8+6,9+6},{1+6,2+6,5+ 6,6 + 6,10 + 6,11 + 6}) = ({6,9,10,1,2,3},{7,8,11,0,4,5}), and we see that the image of minor second 1 is the perfect fifth 7! So 7 appears as a transformed dissonance, see right half of Figure 12. If we had the consonance ξ = 3, then the consonance η = 7 would be a transformed dissonance, i.e. a consonance that is on the other side of the transformed dichotomy (K+6, D+6), while 3 is a transformed consonance. In this way we may move from one consonance to another consonance but move from the 'imaginary' dichotomy's consonant part K+6 to its dissonant part D+6. In our example this would enable us to move from the minor third to the perfect fifth, a movement that we observe in our very first example in Figure 8 from the third to the fourth interval.

The idea here is that we simulate a separation of consonances from dissonances by a 'deformation' of the (K,D) dichotomy that would create imaginary consonances and dissonances within K. We would understand this deformation as the result of a force action (the transformation we used above!) upon (K, D) to make it produce the imaginary dichotomy. In the theory of this approach it can be shown that such deformations in fact strongly relate to what physicists call "local symmetries", which are responsible for force fields.

Does this approach produce the rules which Fux has given to us in his catechism of counterpoint? The answer is: yes, it does. It in particular produces the forbidden parallel of fifths and no other forbidden parallels, see (Mazzola, 2002, chapter 31.4).

Summarizing, we have proposed an extension of the wall of tension by local symmetries, which transport the dichotomic tension of (K,D) into K. But we have much more than this interpretation of tension in terms of symmetries of intervals: We now have five other strong dichotomies, Δ_{64}, Δ_{68}, Δ_{71}, Δ_{75}, Δ_{78} on which we can try to compose contrapuntal music!

This type of creative work is not one of direct compositional work, but one of the invention of new strategies to compose contrapuntal music based upon other concepts of interval consonance. We retrieve the classical theory, but also five new contrapuntal worlds to be creative.

Testing the Extension. We have written model compositions relating to other than the classical Fuxian counterpoint dichotomy, see Figure 13.

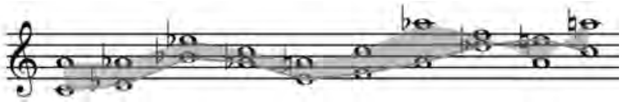

Figure 13. We are taking the major dichotomy Δ_{64} and allow 'consonances' to be the intervals in its I part. This yields the composition as shown.

And Julien Junod (Junod 2010) has implemented a software module in Rubato Composer, which allows to morph a composition written in one dichotomy to a composition in any another dichotomy. Our last example is from the composition *Black Summer* created by Joomi Park for the CD *Passionate Message* (Mazzola, Park, Thalmann 2011, chapter 23). This contemporary composition was created without knowledge about these 'exotic' dichotomies.

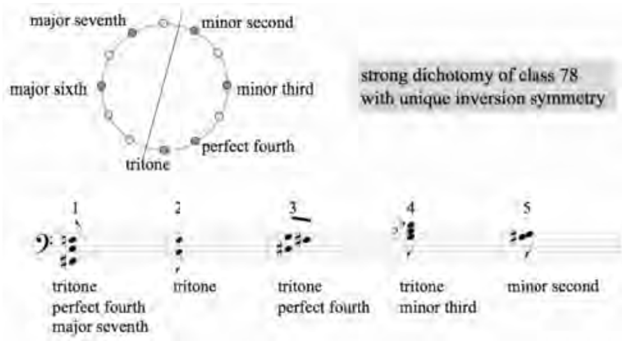

Figure 14. Left: the strong dichotomy of class nr. 78, together with its unique autocomplementarity symmetry. The 'consonances' are the six red pitch classes. Bottom: The numbered intervals of the score of Figure 15 are shown to the bottom.

It is therefore remarkable that she uses the dichotomy of class nr. 78 in her interval selection. The left hand of this homophonic composition is dominated by five of six intervals from the 'consonant' half {1, 3, 5, 6, 9, 11} of the dichotomy, see Figure 14.

Let us look at the score as displayed in Figure 15. These intervals occur in an agreeable and relaxed way as if they were normal consonances. The traditionally consonant unison in the last line then surprisingly appears as a tension! It plays a dissonant role. And it is remarkable that it is 'resolved' to a series of minor seconds in the last measure, since the (consonant) minor second is the interval that corresponds to the unison under the autocomplementarity symmetry of the dichotomy.

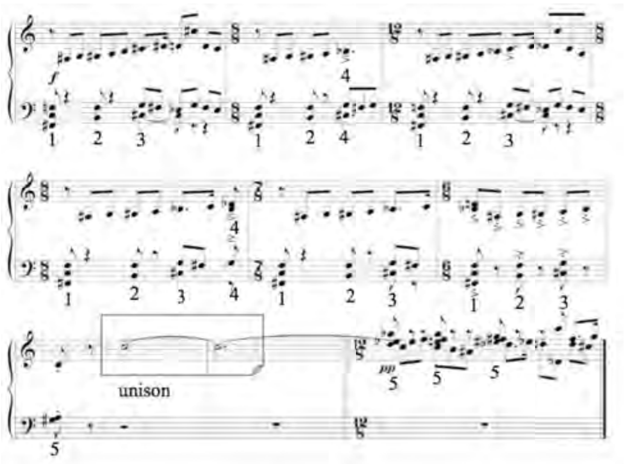

Figure 15. The intervals of the left hand are belonging to the 'consonant' half of the strong dichotomy nr. 78. The numbers of the intervals relate to Figure 14.

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[1] The classical example of this situation is the description of roots of negative numbers that do not exist in the field $\mathbb{R}$ of real numbers. One then considers the ring $\mathbb{R}[Y]$ of all real polynomials in the indeterminate Y and their quotient rings $\mathbb{R}[Y]/(P)$, together with the embeddings: $\mathbb{R}[Y]/(P) \longleftarrow \mathbb{R}$. This is such a characteristic system, and we know that the embedding $\mathbb{R}[Y]/(Y^2+1) \longleftarrow \mathbb{R}$ yields the complex numbers $\mathbb{C} = \mathbb{R}[Y]/(Y^2+1)$ as a solution of the root problem.

[2] The idea of musical forces has been forwarded by famous theorists, such as Arnold Schoenberg. He often speaks about forces between chords in his classical treatise on. He even uses the metaphor of sexual attraction and repulsion between chords.

[Abstract in Korean | 국문 요약]

음악에서의 창조성: 이론, 전략, 작곡 소프트웨어

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이 글에는 수학적 음악 이론과 음악 정보학의 구조유형적(functorial) 접근방법(구조함수Functor란 수학의 범주들 사이에 정해진 구조적 유형을 의미함)이 소개되었다. 이 방법이 음악적 제스처의 이론과 이를 루바토 소프트웨어에서 실행하는데 사용되었는데, 제스처 이론에 입각하여 음악적으로 분석하는 방식으로 설명한다. 그 다음으로 음악적 창조성에 대한 수학적 접근 방법을 소개하였는데, 이 방법은 요네다의 보조 정리를 사용하여 베토벤의 피아노 소나타 op. 109의 3악장에 나오는 여섯 개의 변주곡을 이해하는 것이다. 끝으로, 이러한 수학적 모델과 이를 현대 대위법 이론에 적용시키는 것을 기호학적으로 일반화하면서 이 글은 결론을 맺는다.